Laboratory 11

FREQUENCY RESPONSE AND PASSIVE FILTERS

In this experiment we will analytically determine and measure the frequency response of networks containing resistors, AC source/sources, and energy storage elements (inductors and capacitors).

Given an input sinusoidal voltage, we will analyze the circuit using the frequency-domain method to determine the phasor of output voltage in the ac steady state. The response function is defined as the ratio of the output and input voltage phasors. It is a function of the input frequency and the values of the circuit elements (resistors, inductors, capacitors).

We start with examples of a few filter circuits to illustrate the concept.

**RC Low-Pass Filter:**

Consider the series combination of the resistor R and the capacitor C, connected to an input signal represented by AC voltage source of frequency \( \omega \).

\[
\text{vin}(t) = V_s \cos(\omega t + \theta) \tag{1}
\]

Suppose we are interested in monitoring the voltage across the capacitor. We designate this voltage as the output voltage. We know that it will be a sinusoid of frequency \( \omega \).

Thus,

\[
\text{vout}(t) = V_o \cos(\omega t + \theta_o) \tag{2}
\]

We will now determine expressions for the amplitude \( V_o \) and the phase angle \( \theta_o \). First we convert the network to frequency domain. It is shown in Figure 11.2.
In the above circuit, the voltage source is represented by its phasor and the resistor and capacitor by their impedance. We wish to evaluate the phasor $V_{out}$ for the output sinusoid. Since the three elements are in series, the voltage divider formula can be used and we obtain:

$$V_{OUT} = V_{in} \frac{Z_C}{Z_C + R}$$  \hspace{1cm} (3)

where $V_{in}$ is the phasor of the input voltage. It is given by:

$$V_{in} = V_{se} e^{j\theta}$$ \hspace{1cm} (4)

$$Z_C = \frac{1}{j\omega C}$$ \hspace{1cm} (5)

Manipulation of Equation (3) gives the frequency response as:

$$H(j\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega \tau}$$  \hspace{1cm} (6)

The product $RC$ has units of the inverse of angular frequency. We define $\omega_o = 1/RC$ as a characteristic frequency of the network and write the frequency response as:

$$H(j\omega) = \frac{1}{1 + j\omega/\omega_o}$$  \hspace{1cm} (7)

In other words, we are measuring frequency in units of $\omega_o$.

The sinusoid corresponding to the output voltage can be written as

$$v_{out}(t) = Re\{V_{out} e^{j\omega t}\} = Re\{H(j\omega)V_{in} e^{j\omega t}\} = Re\{V_{se} e^{j\theta} e^{j\omega t}/(1+j\omega/\omega_o)\}$$  \hspace{1cm} (8)

$$v_{out}(t) = \{V_{se} [1+(\omega/\omega_o)^2] \cos(\omega t + \theta) - \tan^{-1}(\omega/\omega_o)\}$$  \hspace{1cm} (9)
Returning to the frequency response, $H(j\omega)$ is a complex number. It has a magnitude and phase. Both depend on the frequency, $R$ and $C$. Thus,

$$H(j\omega) = H \exp(j\theta_H) \quad (10)$$

The magnitude (absolute value) of $H$ is a measure of the ratio of the amplitudes of the output and input voltages. It is given by:

$$H = |H(j\omega)| = \frac{V_o}{V_s} = \frac{1}{\sqrt{1+(\omega/\omega_o)^2}} \quad (11)$$

On the other hand, the phase angle of $H$ measures the difference in the output and input phase angles. It is given by:

$$\theta_o - \theta_I = \theta_H = -\tan^{-1}(\omega/\omega_o) \quad (12)$$

The frequency dependence of the magnitude $H$ is sketched in Figure 11.3.

![Magnitude Response](image)

Fig. 11.3

It can be seen that at low frequencies ($\omega<\omega_o$), $H$ is close to unity. In this frequency range, the network allows effective transmission of the input voltage. For $\omega>>\omega_o$, $H$ becomes very small compared to unity. This means that high frequencies do not get transmitted well by the network. In other words, the network acts as a **low-pass filter**.

The characteristic frequency $\omega_o$ is called the **cut-off frequency**. It is defined as the frequency at which $H$ is equal to $(1/\sqrt{2})H_{max}$. Similarly, the frequency dependence of the phase $\theta_H$ is shown in Figure 11.4. There is negligible phase shift at very low frequencies and approaching $-90^\circ$ at very high frequencies.
The magnitude and phase plots shown in Figures 11.3 and 11.4 are linear. However, in electrical circuits, the frequency range may span several decades. For example, in audio amplifiers, the frequency range of interest is 20 Hz to 20,000 Hz. Similarly, the magnitude of the frequency response may vary over several orders of magnitude. Therefore, linear plots are of little use and the frequency response is represented by **Bode Plots**.

In Bode plots, one plots the magnitude $H$ on the vertical axis, in units of dB, defined by the following equation:

$$H_{\text{dB}} = 20 \log H$$

(13)

On the horizontal axis, the frequency is represented on a log scale. On the log scale, the distance between 10 and 100 rad/s is equal to that between 100 and 1000 rad/s. This is due to the fact that $(\log 100 - \log 10) = (\log 1000 - \log 100)$. You can easily infer that since $(\log 20 - \log 10) = 0.3$, the distance from 10 to 20 is 30% of the distance between 10 and 100.

Figure 11.5 shows the Bode plot of the magnitude and phase of the low-pass filter of Figure 11.1.
At low frequencies, the value of $H_{\text{dB}}$ is close to 0 dB and it is represented by a straight line with zero gradient. At the cut off frequency $H_{\text{dB}}$ drops to $-3$ dB, and at frequencies much larger than the cutoff frequency, the response is accurately represented by a straight line with a slope of $-20$ dB/decade. If we extrapolate the two straight lines, they will intersect at the cutoff frequency. The two lines represent the asymptotic Bode Plots. The maximum error in asymptotic Bode plot for this case is 3 dB, occurring at the cutoff frequency.

Asymptotic Bode plots are very useful in estimating the magnitude $H$ at any frequency fairly accurately. They are easy to sketch since only straight lines are involved. For example, if we wish to know $H$ at a frequency 100 times larger than the cutoff frequency, we get $H_{\text{dB}} = -40$ dB, which gives $H = 0.01$, implying that the amplitude of the output voltage at this frequency is 1% of the amplitude of the input voltage.

When $H$ is smaller than unity, $H_{\text{dB}}$ is a negative number. That means the output voltage amplitude is smaller than the input voltage amplitude and the network attenuates the input signal. Such is the case in the passive low-pass filter considered thus far. We will see later that when active elements such as Op Amps are used, there is usually a net gain and $H_{\text{dB}}$ can be a positive number.

One can design a low-pass filter using an inductor and a resistor, as shown in Figure 11.6. It has characteristics very similar to the RC low-pass filter we analyzed above. In the Prelab you will look at this example.

![Figure 11.6](image-url)
**RC High-Pass Filter**

Suppose that in the network of Figure 11.1, we monitor the output voltage across the resistor as we vary the frequency. It can be shown that

\[
H(j\omega) = \frac{j\omega}{1 + \frac{j\omega}{j\omega_0}}
\]

Where \(\omega_0 = 1/RC\).

The Bode Plot of this filter is shown in Figure 11.7.

![Bode Diagrams](image)

Figure 11.7

It is obvious the network acts as a high-pass filter. The asymptotic Bode plot once again is given by two straight lines. For low frequencies, the slope of the line is +20 dB/decade. The maximum error of 3 dB occurs at the cutoff frequency \(\omega_0\).

A simple passive high-pass filter can also be designed using an inductor and a resistor. (See the prelab).

**Band-Pass Filter**

Consider the series combination of a resistor, an inductor, and a capacitor, as shown in Figure 11.8.
We will monitor the output voltage across the resistor. In frequency domain, we use the voltage divider formula to obtain the phasor for the output voltage.

\[ V_{\text{out}} = V_{\text{in}} \frac{R}{R + j(\omega L - \frac{1}{\omega C})} \]  

(15)

From the above equation, we get the magnitude of the frequency response.

\[ |H(j\omega)| = \frac{R}{R^2 + (\omega L - \frac{1}{\omega C})^2}^{1/2} \]  

(16)

The magnitude of the frequency response is shown in Figure 11.9 for R/L = 1. On the horizontal axis, the frequency has been normalized to \( \omega_0 = 1 \), the resonance frequency given in equation 17.
At very low frequencies, the capacitor has very large impedance, resulting in a low output voltage. Similarly, at very large frequencies, the inductor offers large impedance which results in a drop in the output voltage. However, when the impedances of the capacitor and the inductor cancel each other, the series combination of the two energy-storage elements acts as a short circuit and all the input voltage appears across the resistor \((H = 1)\). This frequency is called the **resonance frequency**.

The resonance frequency is given by

\[
\omega_0 = (LC)^{-1/2}
\]  

(17)

It is seen that the network allows efficient transmission of frequencies in the vicinity of the resonance. This is why it is called a band-pass filter.

Apart from the resonance frequency, the filter is also characterized by its **band width** and \(Q\) (quality factor). The bandwidth and \(Q\) are defined as

\[
BW = \omega_2 - \omega_1
\]

(18)

\[
Q = \frac{\omega_0}{BW}
\]

where \(\omega_1\) and \(\omega_2\) are the two frequencies at which \(H = (1/\sqrt{2})H_{\text{max}}\). It can be shown that for this band-pass filter, \(BW = R/L\). Figure 11.10 shows the Bode plot of the band-pass filter for \(R = 10\ \Omega,\ L = 10\ \text{mH},\ \text{and}\ C = 100\ \mu\text{F}\.\)
Prelab:

Prior to the laboratory do the following:

1. Derive the response function \( \frac{V_{\text{out}}(j\omega)}{V_{\text{in}}(j\omega)} \) for the lowpass RL circuit in Figure 11.6.

2. Derive the response function \( \frac{V_{\text{out}}(j\omega)}{V_{\text{in}}(j\omega)} \) for the highpass RL circuit in Figure 11.11.

3. Derive the response function \( \frac{V_{\text{out}}(j\omega)}{V_{\text{in}}(j\omega)} \) for the bandpass RLC circuit in Figure 11.12.

Note: \( Z_c = \frac{1}{j\omega C} \); \( Z_L = j\omega L \)
Procedure:

Low Pass Filter:

1. Build the circuit in Figure 11.1. Set \( R = 2.2 \, \text{k}\Omega \) and \( C = 0.1 \, \text{uF} \). Use a 4-Vpeak sinusoidal voltage for \( V_{\text{in}} \).

2. Determine the cutoff frequency \( \omega_0 \) for this circuit using circuit analysis.

3. Measure \( V_{\text{outAC}} \) at the cutoff frequency \( \omega_0 \). Additionally, take 5 data points each above and below the cutoff frequency. Make sure to spread out your frequency values. Tabulate your data.

4. Draw a plot of \( H_{\text{dB}} \) vs. frequency for this circuit using the values obtained in step (3). Use Excel or MATLAB to plot the measured values. Compare this plot to the theoretical Bode magnitude plot of the circuit. From the plot determine the value of \( \omega_0 \). Does this value agree with that of step (2)? Comment on any differences.

High Pass Filter:

1. Using the same circuit in Figure 11.1 monitor the voltage across the resistor (R) instead of the capacitance (C).

2. Repeat steps 2-4 from the low pass exercise above.

Band Pass Filter:

1. Build the circuit in Figure 11.12. Set \( R = 470 \, \text{\Omega} \), \( C = 1 \, \text{uF} \), and \( L = 2 \, \text{mH} \). (Note the 2mH inductor was chosen to have a low coil resistance.) Use a 4-Vpeak sinusoidal voltage for \( V_{\text{in}} \).

2. Determine the resonant frequency \( \omega_0 \) for this circuit using circuit analysis.

3. Also determine the theoretical \( \text{Gain} = \frac{V_{\text{outAC}}}{V_{\text{inAC}}} \).

4. Using both channels of the oscilloscope, measure \( V_{\text{inAC}} \) and \( V_{\text{outAC}} \) at the resonant frequency \( \omega_0 \). Hence find the Gain.

5. Additionally, take 5 data points each above and below the resonant frequency \( \omega_0 \). Make sure to spread out your frequency values. Tabulate your data. Do you notice your measurements of Vin change as the frequency changes? If yes, explain. (Hint: consider the equivalent resistance of the function generator)
6. Draw a plot of $H_{dB}$ vs. frequency on a log scale (i.e., a magnitude Bode plot) for this circuit using the values obtained in step (3). Compare this plot to the theoretical Bode magnitude plot of the circuit. From the plot determine the value of $\omega_0$. Does this value agree with that of step (2)? Comment on any differences. Compare the gain at $\omega_0$ to what you expect theoretically. Discuss possible reasons for the differences.

7. What is the bandwidth of this filter?

Band Stop Filter:

Often times it is desired to remove a particular or narrow range of frequencies from a signal. For example, you may want to remove (notch) the 60 Hz line interference from your signal while allowing all other frequencies to pass through undistorted. One solution to this problem is to design a bandstop filter (notch) to remove the unwanted components. The magnitude response may be considered to be the complement of the bandpass response. Figure 11.13 presents the Bode magnitude response of a normalized bandstop filter. Its response function can be expressed as:

$$H(j\omega) = \frac{[(j\omega)^2 + \omega_0^2]}{[(j\omega)^2 + (\omega_0/Q)(j\omega) + \omega_0^2]}$$

Where $Q$ and $\omega_0$ are defined in equations (17) and (18) respectively. For the band stop case, $\omega_0$ is also referred to as the notch frequency.

![Bode Diagrams](image)

Figure 11.13
1. Using the same circuit from the bandpass case, monitor the voltage across the resistor (R) instead of the LC branch. In order to observe the voltage across the resistor do the following: Connect CH1 to node \( a \) and CH2 to node \( b \). Each oscilloscope ground should be connected to the circuit ground. Next, invert CH2 and then add it to the result on CH1. This can be done by making the appropriate settings to a user defined math channel on the oscilloscope i.e. CH1-CH2. This will result in the two voltages being subtracted. Note that the oscilloscope ground should not be connected to either node. Instead, it must be connected to the circuit ground.

2. Determine the notch frequency \( \omega_0 \) for this circuit using circuit analysis.

3. Measure \( V_{outAC} \) at the notch frequency \( \omega_0 \). Additionally, take 5 data points each above and below the notch frequency \( \omega_0 \). Make sure to spread out your frequency values.

4. Draw a plot of \( H_{dB} \) vs. frequency for this circuit using the values obtained in step (3). Compare this plot to the theoretical Bode magnitude plot of the circuit. From the plot determine the value of \( \omega_0 \). Does this value agree with that of step (2)? Comment on any differences.

The Bode Analyzer:

The Bode Analyzer automatically steps through a range of frequencies specified by the user. The analyzer requires that FUNC_OUT be used as the input signal of the circuit. In addition, this input, as well as the ground, must be connected to one of the inputs on the Workbench. Please see Fig. 11.14 for more details.

1. Disable the workbench and close the Function Generator and Oscilloscope panels. Open the Bode Analyzer panel. Please note: This portion of the lab requires moderate changes to the circuit, shown in Figure 11.14.

2. Assemble the band-pass filter from above (Figure 11.12). Be sure that FUNC_OUT is used for \( V_{in} \) but you do not need to start the function generator.

3. Connect \( V_{in} \) to ACH1+ on the Workbench.

4. Connect Ground to ACH1- on the Workbench.

5. Connect the positive node of your output (\( V_{out} \)) to ACH0+ and the negative node to ACH0-.

6. Select appropriate start/stop frequencies in the panel and run the instrument.

7. Does the output match your results from above? Compare this output to your results and the theoretical Bode plot of the magnitude.

8. Repeat this for the band-stop filter.
Design Problem:

1. Derive the frequency response function for the band stop case above and show that it is given by equation (19).

2. Design a circuit to notch out the 60 Hz line frequency with a Q of unity. Use the circuit in Figure 11.12. Part of the problem is to select the appropriate values for R, L, and C. Include a diagram of your circuit.

3. Plot the Bode magnitude response for your circuit.

4. Can the circuit in Figure 11.8 be used as a notch circuit? Explain.